A derivation of ErlangC from first principles (excerpt)

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Contents

1. Introduction

The ErlangC equation is widely used in inbound call centres around the globe. It is deployed within call centres that are characterised as follows: a large number of advisors who can handle every call type, together with random call arrival and random service durations.

It provides an estimate of the number of advisors required to answer a given number of calls and a given average call handling time. It requires two other inputs: the target time to answer a call, together with the target proportion of calls that should be answered within that time.

The equation is derived using principles from a branch of Operational Research techniques called Queueing Theory.

This document includes a full proof of the ErlangC equation, and details of how the equation can be turned into a computable algorithm for insertion into code, for example an Excel macro or Javascript.

2. Queueing Theory Basics

n = number of items in system

λ_n = mean arrival rate of new items when n items are in system
= expected number of arrivals per unit time when n

μ_n = mean service rate when n items are in system
= expected number of items completing service per unit time

$P_n(t)$ = Probability that exactly n items are in the system at time t

s = Number of servers (in parallel channels)

π = Probability that a call arrives to find all servers busy
= Probability that caller has to wait for a server

Note that since there are no services when there is none in the system, then it follows that $\mu_0 = 0$.

Also, if $\lambda_n = \lambda$ is constant for all n, and the mean service rate per busy server is a constant μ, then we can define the following...

$1/\lambda$ = Mean interarrival time (sec)

$1/\mu$ = Mean service time (sec)

We can also define ρ as the utilisation factor for the service facility, defined as the expected fraction of the time the servers are busy:

$$\rho = \frac{\lambda}{s\mu} \qquad (1)$$

A system is in steady state if the number of items in the system does not tend to infinity, ie that in the long term, the system can output contents at a faster rate than others can enter.

If $\rho < 1$, then the system is in steady state.

3. State Probabilities

We start the proof of Erlang by considering a system with random arrivals and random service times.

3.1 Input Behaviour

Given *n* items in system at time *t*, the probability that exactly one arrival will occur during the time interval $(t, t+\delta t)$ is

$$\lambda_n \delta t + o(\delta t) \qquad (2)$$

If we assume that $\lambda_n = \lambda$ (constant) for all *n*, then the system has a random Poisson input. This implies that the number of arrivals in a time interval of length *t* has a Poisson distribution of parameter λt.

$$\Pr(k \text{ arrivals in time interval } t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \qquad (3)$$

The interarrival times have a negative exponential distribution with parameter λ. This verifies $1/\lambda$ as the mean interarrival time.

3.2 Output Behaviour

Given *n* items in system at time t, the probability that exactly one item leaves the system in $(t, t+\delta t)$ is

$$\mu_n \delta t + o(\delta t) \qquad (4)$$

If output is random and mean service rate per busy server is a constant μ, the service time distribution is also negative exponential with parameter μ.

$$\Pr(k \text{ completions of service in time interval } t) = \frac{(\mu t)^k e^{-\mu t}}{k!} \qquad (5)$$

3.3 Input and Output Probabilities

Consider a time period starting at time t with duration δt. Within time period $(t, t+\delta t)$,

$$\begin{aligned} &\text{P(No. arrivals and departures > 1)} &&= \quad o(\delta t) \\ &\text{P(No. arrivals and departures = 0)} &&= \quad 1 - \lambda_n \delta t - \mu_n \delta t + o(\delta t) \qquad (6) \end{aligned}$$

Infinitesimal order $o(\delta t)$ is assumed to be negligible as $\delta t \to 0$.

The probability $P_n(t+\delta t)$ has three components:

(i) $P_n(t)$ * Pr(zero arrivals and departures in $(t, t+\delta t)$)
$= P_n(t)(1 - \lambda_n \delta t - \mu_n \delta t)$

(ii) $P_{n-1}(t)$ * Pr(one arrival and zero departures in $(t, t+\delta t)$)

$= P_{n-1}(t)\lambda_{n-1}\delta t$

(iii) $P_{n+1}(t)*$ Pr(zero arrivals and one departure in $(t, t+\delta t)$)
$= P_{n+1}(t)\mu_{n+1}\delta t$

Combining these components, we get

$$P_n(t+\delta t) = (1-\lambda_n\delta t - \mu_n\delta t)P_n(t) + \lambda_{n-1}\delta t P_{n-1}(t) + \mu_{n+1}\delta t P_{n+1}(t) \qquad (7)$$

If δt now tends to zero (preserving *t* as a constant), the existence of all derivatives $P_n'(t)$ is established and we obtain a system (ξ) of equations as follows.

$$P_n'(t) = \frac{P_n(t+\delta t) - P_n(t)}{\delta t} = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t)$$
$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \qquad (8)$$

A solution to this set of differential equations is available only in the steady state.

3.4 Steady State Probabilities

If steady state exists for the system (ie $\rho = \dfrac{\lambda}{s\mu} < 1$),

$$P_n(t) \to P_n \text{ as } t \to \infty$$
$$\text{ie } P_n'(t) \to 0 \text{ as } t \to \infty$$

$$-(\lambda_n + \mu_n)P_n + \lambda_{n-1}P_{n-1} + \mu_{n+1}P_{n+1} = 0 \qquad (9)$$

and $\lambda_0 P_0 + \mu_1 P_1 = 0$

As a result,

$$P_n = P_0 \left(\frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} \right) \qquad (10)$$

Also, since $\sum_{n=0}^{\infty} P_n = 1 \quad \therefore P_0 + \sum_{n=1}^{\infty} P_n = 1$

Therefore $$P_0 + P_0 \sum_{n=1}^{\infty} \left(\frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} \right) = 1$$

and as a result,

$$P_0 = \left\{ 1 + \sum_{n=1}^{\infty} \left(\frac{\prod_{i=0}^{n-1} \lambda_i}{\prod_{i=1}^{n} \mu_i} \right) \right\}^{-1} \qquad (11)$$

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